Complete Eigenvalue Analysis of Inhomogeneously Dielectric Loaded Two Conductor Guiding Structures

Magdy Z. Mohamed and John M. Jarem

Abstract—**A complete full-wave eigenvalue analysis is presented for inhomogeneously filled guiding structures that support TEM mode. The analysis is based on treating the transverse inhomogeneity of the filling dielectric as a polarization current that excites the corresponding empty guiding system. The problem is formulated to determine the quasi-TEM and higher order modes of the system. The mode propagation constants squared appear as eigenvalues of the problem, and the corresponding eigenvectors represent the expansion coefficients of the field modes.**

The strength of the formulation is verified by its application to the problem of the partially dielectric-filled parallel-plate waveguide. The results of the analysis are compared to the exact solution and a variational solution. The quasi-TEM mode variation versus permittivity and frequency variation is studied. The convergence and dispersion characteristics of the method are presented.

I. Introduction

Omar and Shünemann [1] studied the complex and backward-wave modes in inhomogeneously and anisotropically filled waveguides. In their study, they presented an interesting approach to solve the problem of EM wave propagation in inhomogeneous waveguides. Their eigenvalue formulation was based on using TE and TM modes of an empty waveguide as expansion functions, with which the propagating modes of the corresponding inhomogeneous waveguide could be determined.

In the present paper, the formulation of [1] will be extended to solve for the quasi-TEM mode, as well as higher order modes. of the inhomogeneously dielectric loaded multiconductor guiding systems. A limitation of [1] is that, at present, it only applies to hollow waveguides. The purpose of the present chapter will be to extend this method to multi-conductor systems. This extension will be performed by adding a TEM mode to the TE and TM expansion mode sets that [1] is already using. The extension is non-trivial and the results of the analysis show that a significantly different eigenvalue equation form than that which was presented in [1] must be solved when a TEM mode is added to the expansion set of [1].

The analytical expressions of the eigenmodes of the homogeneous waveguide may be obtained for a class of coaxial rectangular waveguides using the Generalized Spectral Domain (GSD) method of [2]. A more general and accurate method to determine the eigenmodes of the coaxial rectangular waveguide using a multipole method is presented in [3]. The multipole method [3] can be applied to the shielded microstrip considering the inner conductor as a wide flat rectangular strip or as a flattened ellipse of high eccentricity which approximates the shape of the rectangular strip.

II. Analysis

Consider a waveguiding system consisting of two perfect conductors enclosing an inhomogeneous dielectric material with cross section S. See Fig. 1. The longitudinal vector is denoted by a_z and the transverse vector is denoted by $\overrightarrow{r}$. The system is uniform in the

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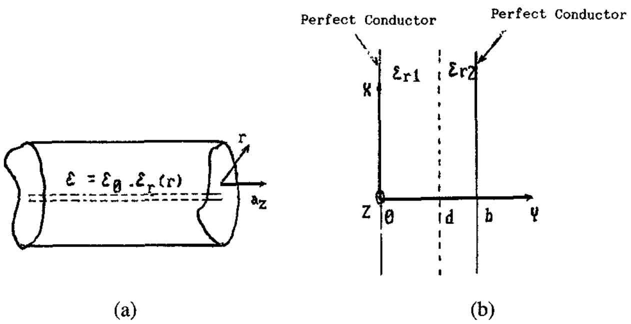


Fig. 1. (a) Inhomogeneously dielectric filled axially wire loaded waveguide. (b) Geometry of a partially filled parallel plate waveguide.

longitudinal direction. The filling dielectric medium is assumed to have transversely dependent relative permittivity $\varepsilon_r = \varepsilon_r(\overrightarrow{r})$ and a constant relative permeability $\mu_r = 1$.

In the present section, an eigenvalue problem will be formulated to solve the problem of full-wave propagation in the system shown in Fig. 1(a). The method presented in [1] will be extended to solve for the quasi-TEM and higher-order modes in the general guiding structure shown in Fig. 1(a). The important point about the present analysis is that it presents a good study of the effect of the inhomogeneity on the TEM mode, which is supported by the corresponding homogeneous system.

The axial electric and magnetic fields of the empty system are defined as in [1]. To be consistent with [1], we further define the electrostatic potential (describing the TEM mode) between the two conductors to be $e_{z0}(\overrightarrow{r})$

$$\nabla_t^2 e_{z0} = 0 \tag{1}$$

The TE and TM modes may be normalized as in [1]. Referring to Harrington [28, pp. 382–383], the TEM mode is normalized according to

$$\int_S (\nabla_t e_{z0})^2 \, dS = 1. \tag{2}$$

Adding the TEM transverse components to the expansion expressions presented in [1], the total transverse fields may be expanded as

$$\begin{aligned}\overrightarrow{E}_{tT} &= e^{-j\beta z}\Bigg\{-A_0\nabla_t e_{z0} + \sum_{n=1}\left[-\frac{A_n}{k_{ne}}\right] \\ &\quad \cdot \nabla_t e_{zn} + \sum_{n=1}\left[\frac{j\omega\mu_0 B_n}{k_{nh}}\right](\hat{a}_z \times \nabla_t h_{zn})\Bigg\} \\ \overrightarrow{H}_{tT} &= e^{-j\beta z}\Bigg\{\sum_{n=1}\left[\frac{-j\beta B_n}{k_{nh}}\right]\nabla_t h_{zn} \\ &\quad + j\omega\varepsilon_0 D_0(\nabla_t e_{z0} \times \hat{a}_z) \\ &\quad + \sum_{n=1}\left[\frac{j\omega\varepsilon_0 D_n}{k_{ne}}\right](\nabla_t e_{zn} \times \hat{a}_z)\Bigg\}\end{aligned} \tag{3}$$

where A_0, A_n, B_n, D_0, and D_n are the series expansion coefficients.

Applying Maxwell's first equation ($\nabla \times \overrightarrow{E} = -j\omega\mu_0\overrightarrow{H}$) to (3) we get

$$D_0 = -\frac{j\beta}{k_0^2}A_0 \tag{4}$$

The total transverse components of the exciting polarization current $\vec{J}$ will then be defined as

$$\vec{J}_{tT} = (\nabla \times \vec{H} - j\omega\varepsilon_0 \vec{E})_t \tag{5}$$

The A_0 and D_0 expansion coefficients are related to the exciting current $\vec{J}$ by

$$e^{j\beta z}(A_0 - j\beta D_0) = \frac{1}{j\omega\varepsilon_0}\int_S \vec{J}_{tT}\cdot\nabla_t e_{z0}\,dS \tag{6}$$

Using (1)–(6) we get the following infinite system of linear equations, which determine the relation between the expansion coefficients A_0, A_n, B_n, D_0, and D_n

$$([I] - k_0^2[S])\underline{D} = j\beta[S]\underline{A} \tag{7a}$$

$$R^e_{00}A_0 + \underline{R}^e_{01}A - j\beta D_0 = -j\omega\mu_0\underline{T}_{01}\underline{B} \tag{7b}$$

$$\underline{R}^e_{10}A^0 + [R^e]\underline{A} - j\beta\underline{D} = -j\omega\mu_0[T]\underline{B} \tag{7c}$$

$$(k_0^2[R^h] - (\Lambda^h])\underline{B} - \beta^2\underline{B} = j\omega\varepsilon_0\underline{T}_{10}A_0 + j\omega\varepsilon_0[T]^t\underline{A} \tag{7d}$$

where $\underline{R}^e_{10}$, and $\underline{T}_{10}$ are column vectors with elements $R^e_{n,0}$, and $T_{n,0}$, respectively; $\underline{R}^e_{01}$ and $\underline{T}_{01}$ are row vectors with elements $R^e_{0,n}$ and $T_{0,n}$, respectively. $\underline{A}, \underline{B}, \underline{D}, [\Lambda^h], [R], [S], [T]$, and $[T]^t$ are defined in [1]. R^e_{00} and the elements of $\underline{R}^e_{01}, \underline{R}^e_{10}, \underline{T}_{01}, \underline{T}_{10}$ are given by

$$\begin{aligned}
R^e_{00} &= \int_S \varepsilon_r(\nabla_t e_{z0})^2\,dS\\
R^e_{0,n} &= \frac{1}{k_{ne}}\int_S \varepsilon_r\nabla_t e_{z0}\cdot\nabla_t e_{zn}\,dS\\
R^e_{n,0} &= R^e_{0,n}\\
T_{0,n} &= \frac{1}{k_{nh}}\int_S \varepsilon_r(\nabla_t e_{z0}\times\nabla_t h_{zn})\cdot\hat{a}_z\,dS\\
T_{n,0} &= T_{0,n}
\end{aligned} \tag{8}$$

It may be noticed for the case of inhomogeneously-filled single conductor waveguides, that the term R^e_{00} and the vectors $\underline{R}^e_{01}, \underline{R}^e_{10}, \underline{T}_{01}$, and T_{10} disappear. And in turn, the whole formulation reduces to that presented in [1].

III. Eigenvalue Problem

Equations (4) and (7a) are used to eliminate D_0 and $\underline{D}$, respectively, from the rest of (7). The following eigenvalue problem results

$$\begin{bmatrix} [S^+]^{-1}[R^+] & j\omega\mu_0[S^+]^{-1}[T^+] \\ -j\omega\mu_0[T^+]^t & (k_0^2[R^h] - [\Lambda^h]) \end{bmatrix} \begin{bmatrix} \underline{A}^+ \\ \underline{B} \end{bmatrix} = \beta^2 \begin{bmatrix} \underline{A}^+ \\ \underline{B} \end{bmatrix} \tag{9}$$

where

$$\underline{A}^+ = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix},$$

$$[R^+] = \begin{bmatrix} R_{00} & R^e_{01} \\ R^e_{10} & [R^e] \end{bmatrix},$$

$$[T^+] = \begin{bmatrix} \underline{T}_{01} \\ [T] \end{bmatrix}, \quad \text{and}$$

$$[S^+] = \begin{bmatrix} \frac{1}{k_0^2} & \underline{O}^t \\ \underline{O} & [k_0^2[I] - [S]^{-1}]^{-1} \end{bmatrix}$$

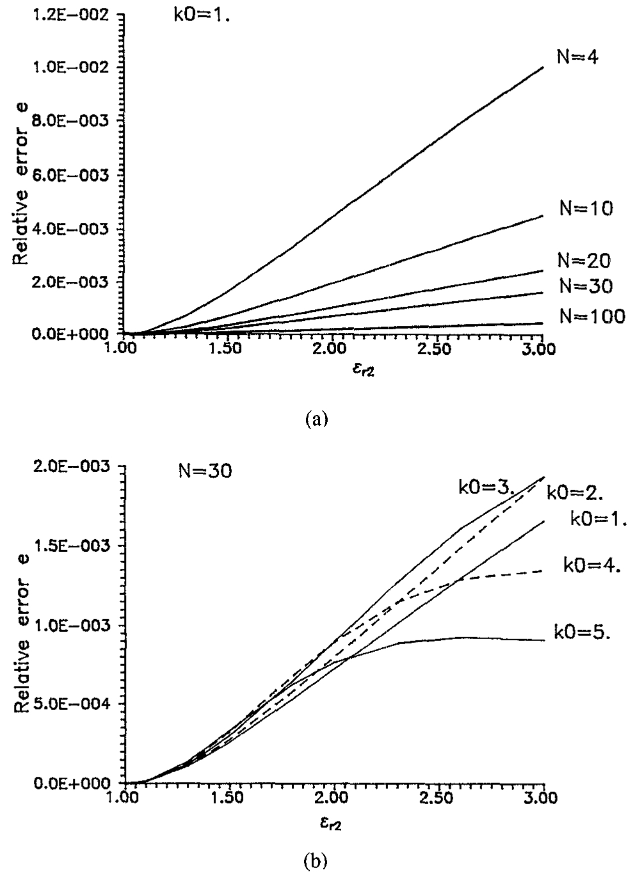


Fig. 2. Plots of the relative error, defined in (15), versus ε_{r2} for the waveguide shown in Fig. 1(b). Where N is the total number of modes considered in the series expansions in (3).

$\underline{O}$ and $\underline{O}^t$ are column and row null vectors, respectively, and $[S]^{-1}$ and $[S^+]^{-1}$ are the inverse matrices of $[S]$ and $[S^+]$, respectively. Again, for the case of single conductor inhomogeneous waveguide the following reductions to the original formulation in [1] occur: $\underline{A}^+$ to $\underline{A}$, $[R^+]$ to $[R]$, $[T^+]$ to $[T]$, and $[S^+]$ to $[k_0^2[I] - [S]^{-1}]^{-1}$.

Note that the final eigenvalue equation (9) is significantly different than the eigenvalue equation [1, Eq. 10]. The difference between both equations is mainly represented by introducing the new submatrices S^+, R^+, and T^+.

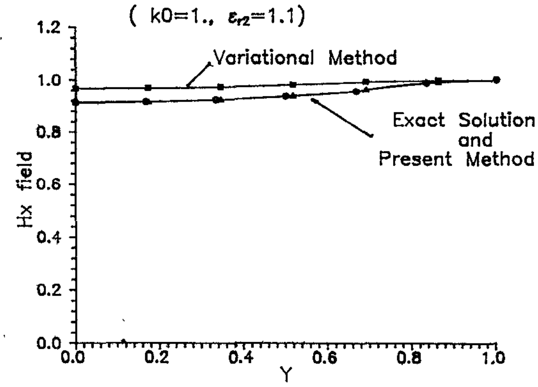
IV. Partially-Filled Parallel-Plate Waveguide

Consider the inhomogeneous parallel-plate waveguide, shown in Fig. 1(b), which is divided into two regions having different relative dielectric constants ε_{r1} and ε_{r2}. The electrostatic potential e_{z0}, the axial electric field e_{zn}, and the axial magnetic field h_{zn} may be defined [5, pp. 133–134] for the empty guide as:

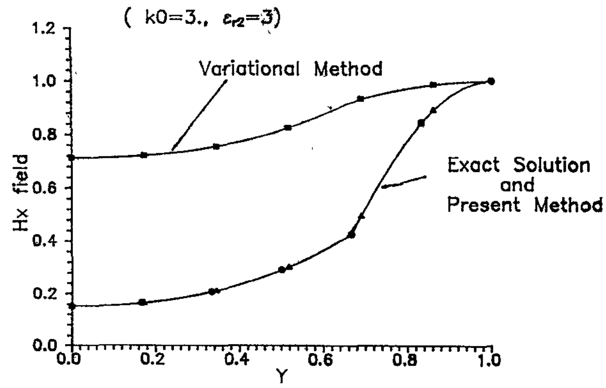
$$\begin{aligned}
e_{z0} &= E_0 y,\\
e_{zn} &= E_n \sin k_{cn} y,\\
h_{zn} &= H_n \cos k_{cn} y
\end{aligned} \tag{10}$$

where $k_{cn} = (n\pi/b)$ and $n = 1, 2, 3, \cdots$. The gradients of these functions are one dimensional and have only one component in y direction.

The two off-diagonal submatrices of the characteristic matrix in (9) will vanish. The eigenvalue problem will be composed of two decoupled parts, the first part being the quasi-TEM and TM fields represented by $\underline{A}^+$, and the second part being the TE field represent



(a)



(b)

Fig. 3. Plots of the magnetic field component H_x versus y for the waveguide shown in Fig. 1(b) ($N = 30$). [Δ—Exact solution, ○ —Present method, □—Variational method].

by $\underline{B}$. Equation (9) is, then, decomposed to

$$[S^+]^{-1}[R^+]\underline{A}^+ = \beta^2 \underline{A}^+ \tag{11}$$

and

$$(k_0^2[R^h] - [\Lambda^h])\underline{B} = \beta^2 \underline{B} \tag{12}$$

The exact solution of the problem, defined in Fig. 1(b), is well-known, e.g., see Harrington [4, pp. 158-191], and may be determined for the TM modes solving the following system of equations:

$$\begin{aligned} k_{y1}^2 + \beta^2 &= \omega^2 \varepsilon_1 \mu_1 \\ k_{y2}^2 + \beta^2 &= \omega^2 \varepsilon_2 \mu_2 \\ (k_{y1}/\varepsilon_1) \tan k_{y1} d & \\ &= -(k_{y2}/\varepsilon_2) \tan[k_{y2}(b-d)] \end{aligned} \tag{13}$$

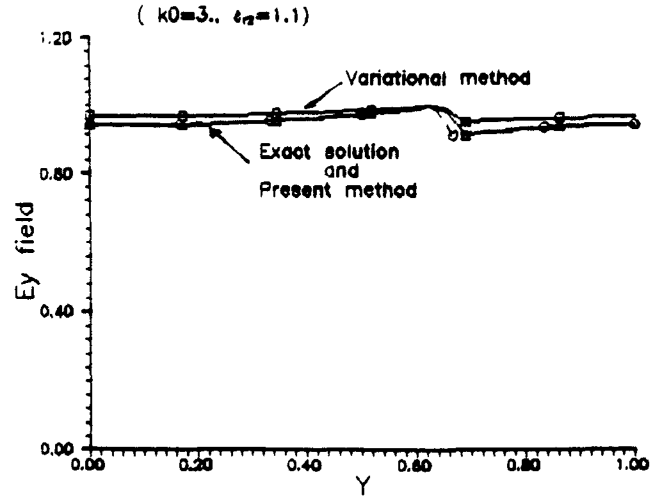
and the axial electric field, in each region, is represented by

$$\begin{aligned} e_{z1} &= c_1 \cos k_{y1} y, \\ e_{z2} &= c_2 \cos[k_{y2}(b-y)] \end{aligned} \tag{14}$$

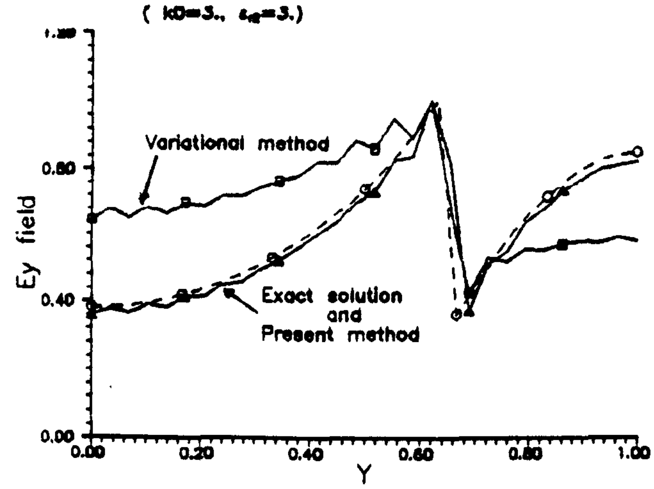
where the suffixes 1 and 2 stand for region number, and k_y is the cut off wavenumber.

Consider the waveguide shown in Fig. 1(b) with $b = 1, d = 2/3$, and $\varepsilon_{r1} = 1$. (11) and (13) are solved on the CRAY supercomputer using well-known IMSL routines: the results are computed, and the relative error e is plotted versus e_{r2} in Fig. 2(a) for different dimensions of the characteristic matrix, N, in (11) when $k_0 = 1$. In Fig. 2(b) the relative error e is plotted vesus ε_{r2} for different values of k_0 when $N = 30$. The relative error e is defined by

$$e = (\beta_{(11)} - \beta_{(13)})/\beta_{(13)} \tag{15}$$



(a)



(b)

Fig. 4. Plots of the electric field component E_y versus y for the waveguide shown in Fig. 1(b) ($N = 30$). [Δ—Exact solution, ○ —Present method, □—Variational method].

where $\beta_{(11)}$ and $B_{(13)}$ are the phase constants resulted from solving (11) and (13), respectively.

From Fig. 2(a), it can be seen that the method converges rapidly for dimensions of matrices in (11) larger than 30, which corresponds to considering only 30 terms in the infinite expansion expression for the fields in (3).

The functions defined in (10) are substituted into (3) to define the transverse fields H_x and E_y, considering only 30 terms in the expansion. In computing the H_x and E_y two different values of ε_{r2} are used, namely $\varepsilon_{r2} = 1.1, 3$. The transverse field components H_x and E_y of the quasi-TEM mode are plotted versus the guide width y in Figs. 3 and 4 when $k_0 = 1$ and 3 respectively. Furthermore, the same fields H_x and E_y, are computed using the variational method [6] and plotted in Figs. 3 and 4. The plots in Figs. 3 and 4 show that the solution resulting from the present method works amazingly well and approximately coincides with the exact solution, even for larger values of ε_{r2}, while the variational solution [6] diverges from the exact one as ε_{r2} increases.

V. Conclusion

A two-conductor system inhomogeneously filled with dielectric is analyzed. The analysis represents a full-wave solution of the problem.

In the analysis, the propagation characteristics of the wave are determined by formulating an eigenvalue problem, where its eigenvalues are the propagation constants squared and the eigenvectors represent directly the expansion coefficients of the propagating fields. The analysis treats the inhomogeneity as a polarization current exciting the corresponding homogeneous guiding system.

The method is used to solve the problem of electromagnetic wave propagation in partially-filled parallel-plate waveguide. The convergence, dispersion characteristics, and accuracy of the method are studied and compared to the results of the variational method [6]. The present method proved to have better accuracy than the variational method [6].

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An Efficient Numerical Procedure Using the Shifted Power Method for Analyzing Dielectric Waveguides Without Inverting Matrices

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***Abstract*—A numerical procedure using the finite-difference scheme and the shifted power method is used to analyze the propagation characteristics of dielectric waveguides. The unique feature of this procedure is that in determining the eigenvalues corresponding to dominant modes no operation as costly as matrix inversion, such as Gaussian elimination, LU decomposition, or tridiagonalization, is invoked. So the proposed procedure is rather efficient in both memory space and computer time. Numerical results of a circular step-index fiber are presented for comparison. Due to its efficiency, the proposed procedure is capable of analyzing coupled waveguides.**

I. INTRODUCTION

Applying the finite-difference or finite-element method to analyzing the propagation characteristics of dielectric waveguides has been investigated for a long time and by many people. These numerical methods render the partial differential equation governing the propagation characteristics of dielectric waveguides to linear simultaneous algebraic equations, which are manageable on a computer. In essence the matrix equation is an eigenvalue problem. Depending on the mathematical formulation, the properties of boundary conditions, and the methods of evaluating the eigenvalues, the eigenvalue problem can be written in several forms, which will be discussed in detail in Section II.

In waveguide theory the important eigenvalues corresponding to dominant guided modes in which one is interested are the largest propagation constants or the lowest frequencies. To calculate these particular eigenvalues several methods have been employed, such as the zero-determinant searching, the inverse power method [1, ch. 10], the subspace iteration method [2] (a variation of the inverse power method using simultaneous iteration), and the method involving tridiagonalization and Sturm sequence [1, chs. 8 and 9]. For evaluating the determinant of a matrix or for inverting a matrix in using the inverse power method one needs to use, for example, Gaussian elimination or LU decomposition. The matrix resulting from applying the finite-difference or finite-element method to a differential equation is banded. The bandwidth grows with the dimension of the problem. For one-, two-, and three-dimensional cases, the value of bandwidth M are of order unity, $O(N^{1/2})$, and $O(N^{2/3})$, respectively, where N is the matrix order. For a band matrix the number of required operations for matrix inversion is of order $O(M^2N)$. The number of required operations for tridiagonalizing a symmetric matrix is of the same order as that of matrix inversion. Thus for the two-dimensional waveguide problem the required computation will go as N^2, when conventional approaches were used.

After analyzing the distribution of eigenvalues of an associated matrix in Section III, we show that the direct power method can be used to calculate the eigenvalues and associated eigenfunctions corresponding to dominant modes by suitably shifting the eigenvalues. The unique feature of this procedure is that no matrix inversion or tridiagonalization is invoked. Thus the proposed procedure is efficient in both computation speed and memory space and is simple in the programming work. A major drawback of the proposed procedure is the slow convergence rate. Some methods to accelerate this rate will be discussed in Section V. Numerical results for circular step-index fiber and coupled rectangular waveguides are presented in Section VI.

II. EIGENVALUE PROBLEMS

Consider a transversely inhomogeneous dielectric waveguide in which a transverse field ψ satisfies the scalar wave equation

$$\nabla_t^2\psi(x,y) + [k_0^2\epsilon(x,y) - \beta^2]\psi(x,y) = 0, \tag{1}$$

where $k_0^2 = \omega^2\mu_0\epsilon_0$, $\epsilon(x,y)$ denotes relative permittivity distribution, and β is the propagation constant in the axial direction. Suppose that dielectric waveguide is cladded by a homogeneous medium with relative permittivity ϵ_1 and the maximum value of the permittivity $\epsilon(x,y)$ is ϵ_2. It is of convenience to express the permittivity $\epsilon(x,y)$ as

$$\epsilon(x,y) = \epsilon_1 + (\epsilon_2 - \epsilon_1)P(x,y), \tag{2}$$

where the profile $P(x,y)$ is zero in the cladding and its maximum value is unity. Using the normalized propagation constant B and normalized frequency V:

$$B = \frac{(\beta/k_0)^2 - \epsilon_1}{\epsilon_2 - \epsilon_1} \tag{3}$$

$$V = k_0 b\sqrt{\epsilon_2 - \epsilon_1}/\pi, \tag{4}$$

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